BASIC RELATIVE INVARIANTS ON THE DUAL CLANS OBTAINED BY REPRESENTATIONS OF EUCLIDEAN JORDAN ALGEBRAS

HIDETO NAKASHIMA and TAKAAKI NOMURA

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We present some detail about what we have written in "added in proof" at the end of our paper published in Kyushu J. Math., **67** (2013).

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1. PRELIMINARIES

Let V be a finite-dimensional real vector space, and Ω a regular open convex cone in V. By regularity we mean that Ω contains no entire line. We denote by $G(\Omega)$ the linear automorphism group of Ω:

$$G(\Omega) := \{g \in GL(V)\ ;\ g(\Omega) = \Omega\}.$$

$G(\Omega)$ is a Lie group as a closed subgroup of $GL(V)$. We say that Ω is *homogeneous* if $G(\Omega)$ acts on Ω transitively.

By Vinberg [7], homogeneous regular open convex cones correspond to non-associative algebras called clans with unit element bijectively up to isomorphisms. We recall here the definition of a clan. A real vector space V with a bilinear product $x \triangle y = L(x)y$ is called a *clan* if the following three conditions are satisfied.

(1) $[L(x), L(y)] = L(x \triangle y - y \triangle x)$ for any $x, y \in V$.

(2) There exists $s \in V^*$ such that $s(x \triangle y)$ defines a positive-definite inner product in V.

(3) Every operator $L(x)$ $(x \in V)$ has only real eigenvalues.

Linear forms s with the property in (2) are called *admissible.*

More generally, what Vinberg [7] has established is the fact that homogeneous regular convex domains correspond bijectively to clans (not necessarily with unit element) up to isomorphisms.

Now we return to the situation in the first paragraph. In order to obtain a clan structure in the ambient vector space V of Ω, we proceed in the following way. First we know that $G(\Omega)$ has a split solvable Lie group (Borel subgroup) H acting simply transitively on Ω. By fixing a point $E \in \Omega$, the simple transitivity of the action yields that the orbit map $H \ni h \mapsto hE \in \Omega$ is a diffeomorphism. Let $\mathfrak{h} := \mathrm{Lie}(H)$. By differentiation at the unit element of H, we obtain a linear isomorphism $\mathfrak{h} \ni T \mapsto TE \in V$. We denote by $L : x \mapsto L(x)$ its inverse map, that is, for every $x \in V$, the operator $L(x) \in \mathfrak{h}$ is the unique operator such that $L(x)E = x$. Making use of these $L(x)$, we define a product in V by $x \triangle y := L(x)y$ $(x, y \in V)$. Then it turns out that V is a clan with unit element E. The clan product is, in general, non-commutative and non-associative.

Now we assume that V is equipped with an inner product $\langle \cdot \,|\, \cdot \rangle$. By the dual cone Ω^* with respect to $\langle \cdot \,|\, \cdot \rangle$ we mean the set defined by

$$\Omega^* := \big\{ y \in V \; ; \; \langle \, x \,|\, y \, \rangle > 0 \quad (\forall x \in \overline{\Omega} \setminus \{0\}) \big\}.$$

If Ω is homogeneous, then so is Ω^*, and we have $G(\Omega^*) = {}^tG(\Omega)$, where ${}^tG(\Omega)$ denotes the totality of the transpose of the operators belonging to $G(\Omega)$ with respect to $\langle \cdot \,|\, \cdot \rangle$. If we have $\Omega = \Omega^*$ with respect to some inner product, then Ω is said to be *selfdual*. Homogeneous selfdual open convex cones are called *symmetric cones*.

As in the book Faraut–Korányi [1], symmetric cones are described by means of Euclidean Jordan algebras. Let us recall here the definition of Euclidean Jordan algebras. First a *Jordan algebra* is a vector space V with a bilinear product xy such that the following (1) and (2) hold for all x, $y \in V$:

$$\text{(1) } xy = yx, \qquad \text{(2) } x^2(xy) = x(x^2y).$$

A Jordan algebra V with unit element e_0 is said to be *Euclidean* if there is a positive-definite inner product $\langle \, x \,|\, y \, \rangle$ with the following property:

$$\langle \, xy \,|\, z \, \rangle = \langle \, x \,|\, yz \, \rangle \quad (\forall x, y \in V).$$

An inner product satisfying this property is called *associative*. In other words, a Euclidean Jordan algebra has an inner product for which every Jordan multiplication operator is selfadjoint. To obtain a symmetric cone Ω from a Euclidean Jordan algebra V, we just take the set of the interior of squares: $\Omega = \mathrm{Int}\,\{x^2 \; ; \; x \in V\}$.

When Ω is a symmetric cone in a Euclidean Jordan algebra V, we know that its linear automorphism group $G(\Omega)$ is a reductive Lie group. By fixing a Jordan frame $c_1, \ldots, c_r$ of V, where r is the rank of V, we have an Iwasawa decomposition $G = KAN$ (in the standard notation) of the connected component G of $G(\Omega)$. The split solvable Lie group $H := AN$ acts on Ω simply transitively. Using this H and the unit element e_0 of V, we also have a clan structure in V.

For later use, we record here the definition of a selfadjoint representation of a Euclidean Jordan algebra. Let V be a Euclidean Jordan algebra with unit element e_0. Let E be a real vector space with inner product $\langle \cdot | \cdot \rangle_E$, and we equip $\mathrm{Sym}(E)$ with the Jordan product $A \circ B := \frac{1}{2}(AB + BA)$. We call a Jordan algebra homomorphism $\varphi : V \to \mathrm{Sym}(E)$ a *selfadjoint representation* of V on E. In what follows we always require that if $\dim E > 0$, then $\varphi(e_0)$ is the identity operator on E.

2. BASIC RELATIVE INVARIANTS

Let $\Omega \subset V$ be a homogeneous regular open convex cone of rank r, and as in the previous section, we take a split solvable subgroup H of $G(\Omega)$ acting on Ω simply transitively.

We say that a function f on Ω is *relatively invariant* (with respect to H) if there exists a one-dimensional representation χ of H such that $f(hx) = \chi(h)f(x)$ holds for any $h \in H$, $x \in \Omega$. The following theorem is fundamental.

THEOREM 2.1 (Ishi [2]). *There exist irreducible relatively invariant polynomial functions $\Delta_1, \ldots, \Delta_r$ on V such that any relatively invariant polynomial function P is expressed in a unique way as*

$$P(x) = c\,\Delta_1(x)^{m_1} \cdots \Delta_r(x)^{m_r} \quad (c = \text{const.}, (m_1, \ldots, m_r) \in \mathbb{Z}^r_{\geqq 0}).$$

We call these polynomials $\Delta_1(x), \ldots, \Delta_r(x)$ the *basic relative invariants* associated to Ω. Theorem 2.2 below characterizes the basic relative invariants. First let us recall that by fixing $E \in \Omega$ we have a clan structure in V with unit element E. Let $R(x)$ be the right multiplication operator by x in this clan, that is, we have $R(x)y := y \triangle x$ $(\forall y \in V)$. In the following, we denote by the capital letter $\mathrm{Det}\, T$ the determinant of an operator T or a (real or complex) matrix T, whereas we write $\det x$ for an element x of a Jordan algebra. We distinguish in this way because the Jordan algebra determinant is not multiplicative in general, and we would like to avoid any misuse.

THEOREM 2.2 ([4]). *The irreducible factors of the polynomial* $\mathrm{Det}\, R(x)$ *coincide with $\Delta_1(x), \ldots, \Delta_r(x)$.*

This theorem immediately gives rise to the following problem.

Problem 2.3. Factorize $\mathrm{Det}\, R(x)$. In other words, express the positive integers $n_1, \ldots, n_r$ in the expression $\mathrm{Det}\, R(x) = \Delta_1(x)^{n_1} \ldots \Delta_r(x)^{n_r}$ in terms of the quantities related to the structure of the clan V.

The row vector $\boldsymbol{n} := (n_1, \ldots, n_r)$ is called the *basic index* of the clan V. If V is the clan which is originally a Euclidean Jordan algebra, then we can give a solution to this problem. The proof is given in [5, Theorem 2.9], though the second author got the formula in 2008.

Let Ω be the symmetric cone of rank r in a simple Euclidean Jordan algebra V. Let us fix a Jordan frame and take the split solvable subgroup $H = AN$ appearing in the corresponding Iwasawa decomposition. H acts on Ω simply transitively, and gives a clan structure to V. Let us consider the right multiplication operators $R(x)$ of the clan V. Let $\Delta_1(x), \dots, \Delta_r(x)$ be the Jordan algebra principal minors obtained by the Jordan frame that we are fixing. These principal minors form the basic relative invariants associated to Ω.

Theorem 2.4. *One has* $\operatorname{Det} R(x) = \Delta_1(x)^d \cdots \Delta_{r-1}(x)^d \Delta_r(x)$, *where* d *is the common dimension of the off-diagonals in the Peirce decomposition of* V. *More directly speaking, one has*
(1) $d = 1$ *for* $V = \mathrm{Sym}(r, \mathbb{R})$,
(2) $d = \dim_{\mathbb{R}} \mathbb{K}$ *for* $V = \mathrm{Herm}(r, \mathbb{K})$ $(\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O})$,
(3) $r = 2$ *and* $d = n - 2$ *in the case of the Lorentz cone in* $\mathbb{R}^n$ $(n \geqq 3)$.

Remark 2.5. Counting the degrees of the polynomials in the formula of the theorem corresponds to the following elementary formula:

$$\tfrac{1}{2} r(r-1)d + r = \big(1 + \cdots + (r-1)\big) \times d + r.$$

3. DEFINING CLANS FROM REPRESENTATIONS OF EUCLIDEAN JORDAN ALGEBRAS

We restrict the contents of this section to the extent that we need in the next section. Let V be a simple Euclidean Jordan algebra of rank r with unit element e_0, and fix a Jordan frame $c_1, \dots, c_r$ of V. The inner product $\langle\, x \,|\, y \,\rangle$ in V is given by $\mathrm{tr}(xy)$, where $\mathrm{tr}(\cdot)$ stands for the trace function of V. Let (φ, E) $(\dim E > 0)$ be a selfadjoint representation of V, and we assume that $\varphi(e_0)$ is the identity operator on E. Then we know that $\varphi(c_1), \dots, \varphi(c_r)$ are mutually orthogonal projection operators of the same rank. Denoting the Peirce decomposition of $x \in V$ by $x = \sum_{i=1}^r \lambda_i c_i + \sum_{1 \leqq j < k \leqq r} x_{kj}$, we set

$$\underline{\varphi}(x) := \frac{1}{2} \sum_{i=1}^{r} \lambda_i \varphi(c_i) + \sum_{1 \leqq j < k \leqq r} \varphi(c_k) \varphi(x_{kj}) \varphi(c_j).$$

We call $\underline{\varphi}(x)$ the *lower triangular part* of $\varphi(x)$. We have $\underline{\varphi}(x) + \underline{\varphi}(x)^* = \varphi(x)$. The point here is that φ is a clan representation of V as well. In other words, we have

$$\varphi(x \bigtriangleup y) = \underline{\varphi}(x)\varphi(y) + \varphi(y)\underline{\varphi}(x)^* \qquad (x, y \in V).$$

Let Q be the symmetric bilinear map $E\times E\to V$ defined through

$$\langle\,\varphi(x)\xi\,|\,\eta\,\rangle_E=\langle\,Q(\xi,\eta)\,|\,x\,\rangle\quad(x\in V,\ \xi,\eta\in E).$$

With the above setup, we now introduce a bilinear product $\triangle$ in the space $V_E:=E\oplus V$ by

$$(\xi+x)\triangle(\eta+y):=\underline{\varphi}(x)\eta+(Q(\xi,\eta)+x\triangle y)\quad(\xi,\eta\in E,\ x,y\in V).$$

Then, it turns out that $(V_E,\triangle)$ is a clan. In fact, defining $s'\in V_E^*$ by

$$s'(\xi+x):=\operatorname{Tr}L(x)\qquad(\xi\in E,\ x\in V),$$

we see that s' is admissible.

The clan thus defined does not have unit element. Hence, we adjoin a unit element e to V_E and get a clan $V_E^0:=\mathbb{R}e\oplus V_E$. In what follows, we put $u:=e-e_0$, and we use the decomposition $V_E^0=\mathbb{R}u\oplus E\oplus V$ for the description of V_E^0. Then the clan product of V_E^0 is

$$\begin{aligned}&(\lambda u+\xi+x)\triangle(\mu u+\eta+y)\\&\qquad=(\lambda\mu)u+(\mu\xi+\tfrac{1}{2}\lambda\eta+\underline{\varphi}(x)\eta)+(Q(\xi,\eta)+x\triangle y),\end{aligned}$$

where $\lambda,\mu\in\mathbb{R}$, $\xi,\eta\in E$ and $x,y\in V$. It is easier to understand V_E^0 by imaging it as the following square matrix, where V_E is imaged as the rectangle matrix $\left(\,E\,\middle|\,V\,\right)$:

$$V_E^0=\left(\begin{array}{ccc|c}\lambda&&&\\&\ddots&&{}^tE\\&&\lambda&\\\hline&E&&V\end{array}\right).$$

We know that Q is Ω-positive, that is, $Q(\xi,\xi)\in\overline{\Omega}\setminus\{0\}$ $(\forall\xi\neq0)$. Then we have the corresponding real Siegel domain $D(\Omega,Q)$ defined by

$$D(\Omega,Q):=\left\{\xi+x\in V_E\ ;\ x-\tfrac{1}{2}Q(\xi,\xi)\in\Omega\right\}.$$

The homogeneous open convex cone corresponding to the clan V_E^0 is described as

$$\Omega^0=\left\{\lambda u+\xi+x\in V_E^0\ ;\ \lambda>0,\ \lambda x-\tfrac{1}{2}Q(\xi,\xi)\in\Omega\right\}.$$

In other words, Ω^0 is the open convex cone generated by the origin of V_E^0 and the real Siegel domain $D(\Omega,Q)$ embedded in the hyperplane $\lambda=1$ in V_E^0.

4. THE DUAL CLAN OF V_E^0

We define an inner product $\langle\,\cdot\,|\,\cdot\,\rangle^0$ in $V_E^0 = \mathbb{R}u \oplus E \oplus V$ by
$$\langle\, \lambda u + \xi + x \,|\, \lambda' u + \xi' + x' \,\rangle^0 = \lambda\lambda' + \langle\, \xi \,|\, \xi' \,\rangle_E + \langle\, x \,|\, x' \,\rangle.$$
The dual cone $(\Omega^0)^*$ of Ω with respect to this inner product is the set
$$(\Omega^0)^* := \big\{ v \in V_E^0 \,;\, \langle\, v \,|\, v' \,\rangle^0 > 0 \ \text{ for all } v' \in \overline{(\Omega^0)} \setminus \{0\} \big\}.$$
Then the clan product $\triangledown$ of V_E^0 associated to $(\Omega^0)^*$ is given by $v \triangledown v' = {}^t(L_v^0)v'$, where L_v^0 is the left multiplication operator by v in V_E^0, and ${}^t(L_v^0)$ is the transpose of L_v^0 with respect to the inner product introduced above. The clan $(V_E^0, \triangledown)$ obtained in this way is called the *dual clan* of V_E^0.

PROPOSITION 4.1. *The right multiplication operator $R^{\triangledown}_{\lambda u+\xi+x}$ in the dual clan $(V_E^0, \triangledown)$ is written as an operator matrix on $V_E^0 = \mathbb{R}u \oplus E \oplus V$ in the following way:*
$$R^{\triangledown}_{\lambda u+\xi+x} = \begin{pmatrix} \lambda & \langle\,\cdot\,|\,\xi\,\rangle_E & 0 \\ \frac{1}{2}\xi & \varphi(x) & \underline{\varphi}(\cdot)^*\xi \\ 0 & 0 & R_x^{\triangledown_V} \end{pmatrix},$$
where $R_x^{\triangledown_V}$ denotes the right multiplication operator of V which is turned into the dual clan with multiplication $\triangledown_V$.

Considering the determinant of the operator matrix in Proposition 4.1, we obtain
$$\operatorname{Det} R^{\triangledown}_{\lambda u+\xi+x} = \big(\operatorname{Det} R_x^{\triangledown_V}\big) \operatorname{Det} \begin{pmatrix} \lambda & \langle\,\cdot\,|\,\xi\,\rangle_E \\ \frac{1}{2}\xi & \varphi(x) \end{pmatrix}.$$
Let $\Delta_1^*(x), \ldots, \Delta_r^*(x)$ be the Jordan algebra principal minors of $x \in V$ associated to the Jordan frame $c_r, \ldots, c_1$ obtained by reversing the order of the original Jordan frame $c_1, \ldots, c_r$. Then Theorem 2.4 says
$$\operatorname{Det} R_x^{\triangledown_V} = \Delta_1^*(x)^d \cdots \Delta_{r-1}^*(x)^d \Delta_r^*(x).$$
Hence, we obtain
$$\begin{aligned} \operatorname{Det} R^{\triangledown}_{\lambda u+\xi+x} &= \Delta_1^*(x)^d \cdots \Delta_{r-1}^*(x)^d \Delta_r^*(x) \operatorname{Det} \begin{pmatrix} \lambda & \langle\,\cdot\,|\,\xi\,\rangle_E \\ \frac{1}{2}\xi & \varphi(x) \end{pmatrix} \\ &= \Delta_1^*(x)^d \cdots \Delta_{r-1}^*(x)^d \Delta_r^*(x) \big(\lambda \operatorname{Det} \varphi(x) - \tfrac{1}{2}\langle\, {}^{\mathrm{co}}\varphi(x)\xi \,|\, \xi \,\rangle_E\big), \end{aligned}$$
where ${}^{\mathrm{co}}T$ stands for the cofactor operator of an operator T. If T is invertible, then we have ${}^{\mathrm{co}}T = (\operatorname{Det} T)T^{-1}$. Thus, if T is positive-definite, then so is ${}^{\mathrm{co}}T$.

PROPOSITION 4.2. *Let $v = \lambda u + \xi + x \in V_E^0$. Then one has*
$$v \in (\Omega^0)^* \iff x \in \Omega \ \text{ and } \ \lambda > \tfrac{1}{2}\langle\, \varphi(x)^{-1}\xi \,|\, \xi \,\rangle_E.$$

Remark 4.3. The condition in Proposition 4.2 is what Rothaus [6] called the extension of Ω by the representation of φ. Here by a representation in the sense of Rothaus we mean a representation of Ω, which is defined to be a linear map $R : V \to \mathrm{Sym}(E)$ such that for each $x \in \Omega$, the symmetric operator $R(x)$ is positive-definite, and there exists a subgroup H_0 acting transitively on Ω satisfying

$$\forall h \in H_0,\ \exists T \in GL(E) \text{ with } R(hv) = TR(v)\,{}^tT \quad (\forall v \in V).$$

A Jordan algebra representation is a representation of the corresponding symmetric cone. More generally, a clan representation is also a representation of the corresponding homogeneous open convex cone by Ishi [3].

Now by ([1], Proposition IV.4.2), we have $\mathrm{Det}\,\varphi(x) = (\det x)^{N/r}$ for $x \in V$, where $N := \dim E$. Note that N is always a multiple of r by the same proposition. Let ${}^{\mathrm{co}}x$ be the cofactor element of $x \in V$. If x is invertible, then we have

$${}^{\mathrm{co}}x = (\det x)x^{-1}.$$

In general, $x \mapsto {}^{\mathrm{co}}x$ is a polynomial map of degree $r-1$ defined through the Jordan algebra version of the Cayley–Hamilton theorem. If x is invertible, then we have ${}^{\mathrm{co}}\varphi(x) = (\det x)^{\frac{N}{r}-1}\varphi({}^{\mathrm{co}}x)$ by the above. Hence, we obtain the following proposition.

Proposition 4.4. *One has the following irreducible factorization:*

$$\lambda\,\mathrm{Det}\,\varphi(x) - \tfrac{1}{2}\langle\, {}^{\mathrm{co}}\varphi(x)\xi \mid \xi\,\rangle_E = (\det x)^{\frac{N}{r}-1}\big(\lambda \det x - \tfrac{1}{2}\langle\, \varphi({}^{\mathrm{co}}x)\xi \mid \xi\,\rangle_E\big).$$

Therefore we arrive at the following theorem.

Theorem 4.5. *The basic relative invariants $P_j(v)$ associated to $(\Omega^0)^*$ are given by*

$$\begin{aligned} P_j(\lambda u + \xi + x) &= \Delta_j^*(x) \qquad\qquad (j = 1, \dots, r),\\ P_{r+1}(\lambda u + \xi + x) &= \lambda \det x - \tfrac{1}{2}\langle\, \varphi({}^{\mathrm{co}}x)\xi \mid \xi\,\rangle. \end{aligned}$$

Remark 4.6. By the formula given in Theorem 4.5, it is clear that $\deg P_j(v) = j$ $(j = 1, \dots, r, r+1)$. Moreover, contrarily to the case of Ω^0 (see [5], Theorems 5.4 and 5.8), the description of the basic relative invariants is free from the classification of simple Euclidean Jordan algebras, and also free from the division into the cases according to the extent of non-regularity of the representation φ.

If $x \in V$, we have $\det x = \Delta_r^*(x)$. Thus, the answer to Problem 2.3 in the case of $(\Omega^0)^*$ is

$$\mathrm{Det}\,R_v^{\triangledown} = P_1(v)^d \cdots P_{r-1}(v)^d P_r(v)^{\frac{N}{r}} P_{r+1}(v) \qquad (v \in V_E^0).$$

We now comment on the individual cases.

(1) The Hermitian cases. In this case we have $V = \mathrm{Herm}(r, \mathbb{K})$ ($r \geqq 3$ and $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$), and any selfadjoint representation φ is given as

$$E = \mathrm{Mat}(r \times p,\, \mathbb{K}), \qquad \varphi(x)\xi = x\xi \quad (x \in V,\, \xi \in E).$$

We have $Q(\xi, \eta) = \frac{1}{2}(\xi\eta^* + \eta\xi^*)$.

THEOREM 4.7. $(\Omega^0)^*$ *is linearly equivalent to the following* Ω' *contained in the space* $\mathrm{Herm}(rp + 1,\, \mathbb{K})$*:*

$$\Omega' := \left\{ Y = \begin{pmatrix} \mu & \eta^* \\ \eta & y \otimes I_p \end{pmatrix} \gg 0\,;\ \begin{matrix} \mu \in \mathbb{R}, \quad \eta \in \mathbb{K}^{rp} \\ y \in \mathrm{Herm}(r, \mathbb{K}) \end{matrix} \right\},$$

where $T \gg 0$ *means that* T *is positive-definite. We remark that* $\mathbb{K}^{rp}$ *is not considered as a space of matrices but as a space of column vectors of size* rp*.*

THEOREM 4.8. *The basic relative invariants* $P_j(Y)$ *associated to* Ω' *are given by*

$$P_j(Y) = \Delta_j^*(y)\ (j = 1, \dots, r), \quad P_{r+1}(Y) = \mu \det y - \eta^*({}^{\mathrm{co}}y \otimes I_p)\eta,$$

where ${}^{\mathrm{co}}y$ *denotes the cofactor matrix of* y*, and in the case of* $\mathbb{K} = \mathbb{H}$*, it is considered in the Jordan algebra* $\mathrm{Herm}(r, \mathbb{H})$*.*

Remark 4.9. We emphasize here that if $p > 1$, then Ω' is not a symmetric cone. The basic relative invariants obtained by Theorem 4.8 generalize in a systematic way the example given in [4] for $r = 2$, and $\mathbb{K} = \mathbb{R}$.

(2) The Lorentzian case. In this case the Jordan algebra that we started with can be identified with the linear part $V = \mathbb{R}e_0 \oplus W$ of the Clifford algebra $\mathrm{Cl}(W)$ obtained from a Euclidean vector space W. The linear map $x \mapsto {}^{\mathrm{co}}x$ coincides with the restriction to V of the automorphism $x \mapsto \widetilde{x}$ of $\mathrm{Cl}(W)$ that extends the isometry $w \mapsto -w$ of W.

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Kyushu University,
Graduate School of Mathematics,
744 Motooka,
Nishi-ku Fukuoka 819-0735,
Japan
h-nakashima@math.kyushu-u.ac.jp

Kyushu University,
Faculty of Mathematics,
744 Motooka,
Nishi-ku Fukuoka 819-0735,
Japan
tnomura@math.kyushu-u.ac.jp